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On the physical realizability of singular structural systems

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Abstract

Analytical models of structural systems allow for exceptional, singular geometric configurations, characterized by rank deficiency of the equilibrium and kinematic matrix. The feasibility of physical and numerical realization of such configurations depends on the type of singularity–generic vs. nongeneric. It turns out that some interesting, theoretically predicted and thoroughly studied, types of singular configurations (systems with simultaneous statical and kinematic indeterminacy; unprestressable first-order mechanisms; all higher-order mechanisms; singular configurations of finite mechanisms; and kinematically mobile closed polyhedral surfaces) are nongeneric, hence, physically unrealizable and noncomputable (except for exact or symbolic calculation). Thus, in spite of their sometimes remarkable theoretical features, these systems and configurations are just purely formal constructs. Moreover, their attempted implementation would produce a generic prototype with 'essentially' different properties, including structural response. A few of the somewhat unexpected implications of this observation are discussed and a complete set of analytical criteria for the four statical-kinematic types of realizable structural systems is presented. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

For the purpose of statical-kinematic analysis, a structural system is considered as an assembly of perfectly rigid components linked by ideal positional constraints. There exist four, and only four, statical-kinematic types of systems — two ordinary (geometrically invariant and variant), and two singular (quasi-invariant and quasi-variant). The necessary and sufficient analytical criteria for each type of system stem from the pertinent set of simultaneous constraint equations,

$$F^i(X_1,\cdots,X_n,\cdots,X_N;C_i) = 0,\ i = 1,2,\cdots,C. \tag{1}$$

The C constraint functions F^i relate the N generalized coordinates, X_n, to the geometric parameters, C_i, of the system (e.g., known linear and angular sizes of the structural members). At least one solution to the constraint equations, $X_n = X_n^0$, is assumed to be known and is taken as the reference geometric

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configuration. Further investigation requires expanding the functions F^i into power series at the solution point $X_n = X_n^0$:

$$F^i_n x_n + (1/2!)F^i_{mn} x_m x_n + \cdots = 0, \; m,n = 1,2,\cdots,N. \tag{2}$$

Here x_n are infinitesimal increments of the respective coordinates (that is, virtual displacements of the system) and a repeated index denotes summation over the indicated range.

Consider the linearized constraint equations,

$$F^i_n x_n = 0, \quad F^i_n \equiv \partial F^i / \partial X_{n|0}, \tag{3}$$

whose matrix is the constraint function Jacobian matrix at X_n^0. The matrix rank being $r = N$ is a necessary and sufficient criterion of a geometrically stable (invariant) system; according to (3), its virtual displacements and, the more so, kinematic displacements are zero. A system with $r < N$ is underconstrained. If, in addition, $r < C$, the underconstrained system is in a singular geometric configuration where it is statically indeterminate to the degree $S = (C - r)$; here and below, the statically possible self-stress is assumed comprehensive (comprising all structural members).

At $r < N$ Eq. (3) can be solved in terms of properly selected $V = (N - r)$ virtual displacements chosen as independent; each of them defines an independent virtual displacement mode. The existence of V nontrivial solutions to Eq. (3) indicates V-th degree of virtual indeterminacy of the system and its infinitesimal mobility. The latter is necessary and, almost always, sufficient for kinematic indeterminacy, i.e., finite mobility. Kinematic displacements are nontrivial solutions of the nonlinear constraint Eqs. (1) or (2); the number of such solutions, $K \leq V$, is the degree of kinematic indeterminacy of the system. In contrast to K, which is a global parameter of the system, the degrees of virtual, V, and statical, S, indeterminacies are local, characterizing only a given geometric configuration of the system, but not the system itself.

It may happen that, in spite of $V > 0$, the given solution $X_n = X_n^0$ is an isolated point in the configuration space, so that $V > K = 0$. Then the system is only virtually mobile but kinematically immobile (rigid) and has a unique geometric configuration. Such exceptional underconstrained systems are singular and belong to one of the two existing degenerate types — quasi-invariant if topologically adequate, with the rank $r < r_{\max} = N$, and quasi-variant if topologically inadequate, with $r < r_{\max} < N$ ($r_{\max}$ is the maximum rank of the Jacobian matrix attained in the process of constraint variations, e.g., bar length changes). Both types lack kinematic mobility, yet admit infinitesimal first-order displacements at the expense of second- or higher-order elongations of structural members. Defining and, even more so, evaluating the order of infinitesimal mobility proved controversial (Tarnai, 1989; Kuznetsov, 1991; Connelly and Servatius, 1994).

A general method for evaluating the order of infinitesimal mobility has been suggested by Koiter (see Tarnai, 1989). Although the problem is purely geometric, the method takes advantage of the nonlinear theory of elastic stability. Indeed, stability implies resistance to perturbations; in the absence of external loads and initial member forces, the elastic resistance of structural members is the only possible restoring factor in the model. Thus, elastic stability rules out kinematic displacements (they do not produce any strains), assuring kinematic immobility of the system. The close relation between the issue of infinitesimal mobility and the theory of elastic stability explains the lack of general, necessary and sufficient, analytical criteria and algorithms in both areas.

The results obtained for structural systems with bilateral constraints (typically, bars), have been extended to systems involving unilateral constraints (Kuznetsov, 1972, 1991). Such a system is modelled as an assembly of bars, wires and struts, with inequalities replacing equations as constraint conditions for the latter two types of structural members. The Minkowski–Farkas theorem from the theory of linear inequalities in conjunction with the concept of prestressability (the possibility of a stable self-stress) play a pivotal role in the statical-kinematic analysis of these systems. A concise mathematical compendium of the current state of the art has been presented recently by Connelly and Whiteley (1996).

This paper deals with physical and numerical realization of underconstrained structural systems as well as all systems involving unilateral constraints. Only the so-called generic systems and configurations are realizable, both numerically and physically, and can be constructed as real, material systems. (In what follows, the term real is taken to mean realizable). On the other hand, for reasons discussed in the next section, nongeneric systems and configurations are unrealizable and noncomputable; they represent purely formal, theoretical constructs. Realizability of a model sometimes has unexpected theoretical and practical implications. For example, analytical criteria for each of the four possible statical-kinematic types of real systems are not only different from, but, surprisingly, are simpler than, the current criteria, oblivious to the notion of realizability. A complete set of necessary and sufficient analytical criteria for the four types of real systems is presented below, along with a few of their somewhat unexpected consequences.

The determinative qualification of an analytical model as generic or nongeneric is based on the fundamental mathematical concept of structural stability.

2. Structural stability and statical-kinematic analysis

Structural stability (no relation to structures or mechanics) is a mathematical concept introduced by Andronov and Pontriagin (1937). It stems from the fact that the exact values of parameters of a real system can never be known. Therefore, a basic requirement of any physically meaningful analytical model must be that minute changes in the nominal values of the parameters, as a rule, do not produce any abrupt, 'essential', change in the system behavior. Models satisfying this requirement are called structurally stable; only such models can be meaningful, realizable, and observable as physical phenomena and systems.

A mathematical formulation of the concept reads (see, e.g., Jackson, 1989):

A system of equations (a model) is structurally stable if any sufficiently small change in the model parameters does not result in an 'essential' change in the solutions of the system.

The concept of structural stability underlies an associated notion of computability, known as the Fredkin postulate:

'There is a one-to-one mapping between what is possible in the real world, and what is theoretically possible in the digital simulation world', and the corollary 'That which cannot, in principle, be simulated on a computer, cannot be part of physics.'

The link between computability and structural stability is in that the unavoidably finite precision of computing and, especially, of input data amounts to small perturbations of system parameters. Only for a structurally stable system this does not result in an 'essential' change in the solutions, thus making meaningful computing feasible. In short, structural stability is a prerequisite to both physical and numerical realizability of any model.

Implementing the concept of structural stability requires defining an 'essential' change in the system. Taking a clue from nonlinear dynamics, it has been proposed (Kuznetsov, 1999) to employ for this purpose a notion of *virtual modal equivalence* (topological equivalence of virtual displacement modes in the adjacent configurations being compared). Thus, the criterion of structural stability is virtual modal equivalence of the original and slightly perturbed geometric configurations of the system. According to this criterion, the two ordinary (nonsingular) statical-kinematic types of systems, invariant ($r=N$) and variant ($N > r=C$), are structurally stable. For the purpose of this discussion, the terms 'structurally stable' and 'generic' are equivalent; the latter term is defined and used in the context of framework rigidity in (Graver et al., 1993).

Singular geometric configurations are defined as those with rank-deficient Jacobian (and equilibrium) matrix,

$$N > r < C \tag{4}$$

This condition defines singular configurations both of kinematically immobile, i.e., quasi-invariant and quasi-variant systems ($V > K = 0$) and of kinematically mobile, variant systems ($V > K > 0$), like a three-bar mechanism in a dead-center position. It turned out that all these configurations are structurally unstable, hence, nongeneric: adjacent nonsingular configurations ($r = C$) have topologically different virtual displacement modes as well as a different number, $V = N - r$, of such modes. This 'essential' change in the system properties is produced by a small constraint variation, including, if feasible, a trivial variation, i.e., one associated with a kinematic (inextensional) motion. Thus, all singular configurations must be physically unrealizable and, moreover, noncomputable. This is, indeed, the case, with just one additional stipulation regarding elastic behavior of certain quasi-invariant and quasi-variant systems.

Some, but not all, of these systems are prestressable, that is, capable of admitting prestress (a state of stable self-stress). In a real system, prestress produces elastic strains which can be accounted for by retracting the above idealization of the system material as perfectly rigid. The state space of the system is expanded to include parameters pertinent to elastic behavior. Within the expanded state space, structural stability (the absence of 'essential' changes in the system under small perturbations) is equivalent to elastic stability (resistance to small perturbations). Prestress and associated elastic strains override unavoidable geometric imperfections; in fact, the resulting singular configuration is engendered by statics, not by the infeasible exact geometry. Thus, prestress of finite magnitude entails structural stability; hence, prestressed singular systems (both quasi-invariant and quasi-variant) are structurally stable. Of course, this stability is only local, confined to a finite vicinity of the reference state and dependent on the prestress magnitude; sufficiently large perturbations (say, thermal expansions or support settlements) may overcome elastic strains and produce 'essential' changes in the system configuration or behavior.

The above example of a system structurally unstable within a given state space but stable within an expanded space, is quite typical. A homogeneous rigid sphere on a rigid plane, as any system in neutral equilibrium, is structurally unstable; accounting for the material inhomogeneity or elastic properties makes it (locally) stable. A pencil standing on its sharpened end can be stabilized dynamically and becomes structurally stable within the appropriately expanded state space.

As to unprestressed and, the more so, unprestressable singular configurations of structural systems, all of them are structurally unstable. If realization (either physical or numerical) of such a configuration is attempted, the obtained real system reverts to a pertinent generic, structurally stable type. The latter is determined by relations among the numbers N and C (the topological attributes of the system) and the fully restored Jacobian matrix rank, $r_{\max}$ (the geometric attribute of an adjacent nonsingular configuration). In particular, theoretical systems with simultaneous global statical and kinematic indeterminacy (singular finite mechanisms), upon realization become either geometrically invariant or, if prestressed, quasi-invariant. Systems with higher-than-first order infinitesimal mobility in reality revert to one of the two ordinary types or, if prestressed, to the related generically singular type, but only with first-order mobility.

The concept of structural stability is even more important in the analysis of systems involving unilateral constraints (wires and struts). The fact is that mobility of systems with unilateral constraints is controlled by constraint counteraction. This is expressed analytically by a constraint inequality that negates the feasible domain of the configuration space determined by all of the remaining simultaneous constraint equations and inequalities. An analytical criterion of such a counteracting linear inequality is obtained by reversing the Minkowski-Farkas criterion for a consequence inequality. Applying this criterion requires knowing the current effective model of the system, specifically, the state (engaged or disengaged) of each unilateral constraint; indeed, only engaged unilateral constraints can develop internal forces, whereas disengaged constraints are force-free, making their presence irrelevant for the current (local) effective model.

However, engagement of a unilateral ($\leq$) constraint means satisfying the boundary equality and is structurally unstable: a small geometric imperfection causes disengagement and 'essential' changes in the system response. The only way to attain structural stability is to ensure constraint engagement by statical means, which, in the absence of external loads, amount to prestressing. This is in line with the above argument concerning prestress and elastic strain as a condition for structural stability of quasi-invariant and quasi-variant systems. Furthermore, for systems with unilateral constraints, this argument applies to geometrically invariant systems as well, since the required constraint counteraction presumes their engagement. Hence, a real system with unilateral constraints can be invariant only if prestressed. In fact, all three kinematically immobile types of real systems with unilateral constraints must be prestressed and, vice versa, a prestressed system with unilateral constraints is kinematically immobile and is a tensegrity system (Kuznetsov, 1991).

This conclusion resonates with the following description given by Buckminster Fuller (1980), a college dropout who declared 'intuition being the key to thinking': 'The tensegrity mast demonstrates the use of tension and compression within the same structure,... — discontinuous compression/continuous tension — illustrating tensional integrity or tensegrity.'

This original description of tensegrity (with integrity implying rigidity, or uniqueness of geometric configuration) is meaningful and illuminating. In contrast, the common mathematical usage of this borrowed term, defining a tensegrity framework as *any* assembly of bars, wires and struts (even one without tension or integrity!), is not only counterintuitive, but outright misleading.

3. Necessary and sufficient analytical criteria for the four types of real structural systems

The necessary and sufficient analytical criterion of a geometrically invariant system with bilateral constraints is a full rank of the constraint Jacobian matrix, $r = N$; such a system is structurally stable, that is, insensitive to sufficiently small variations in member shapes and sizes. For a system involving unilateral constraints, the existing criterion of geometric invariance is based on Minkowski–Farkas theorem. It verifies constraint counteraction within the linearized set of constraint inequalities, thus reducing the configuration space to a single point. In statical terms, this condition requires statical indeterminacy and prestressability. To adapt the existing analytical criterion to real systems, recall that prestressability alone is not enough; actual prestress of finite magnitude must be present to render the configuration structurally stable and generic.

Analytical criteria for infinitesimally mobile systems, both quasi-invariant and quasi-variant, are more intricate. In terms of constraint equations (1), the necessary and sufficient criterion for a system with only infinitesimal mobility (an infinitesimal mechanism) involves two requirements:

1. The rank of the constraint Jacobian matrix is $r < N$ (at $r = N$ the system is invariant); and
2. The given solution of constraint equations is an isolated point in the configuration space.

There are no general analytical means verifying whether or not the second requirement is met. The above mentioned idea of Koiter just shifts the challenge onto the theory of elastic stability, but the latter also lacks universal means for stability check. As a result, little progress has been made since the following particular condition was formulated by Kötter (1912).

Without external loads, equilibrium equations in unknown constraint reactions, Λ_i, are

$$F^i_n \Lambda_i = 0 \tag{5}$$

For a system in a singular configuration, $r < C$, homogeneous Eqs. (5) admit $S = (C - r)$ independent nontrivial solutions Λ_{ik} ($k = 1, 2,..., S$), each representing a statically possible state of self-stress. $S \geq 0$ is the degree of statical indeterminacy of the configuration; a statically determinate configuration is nonsingular and belongs to one of the two ordinary types—geometrically invariant (at $r = N$) or variant (at $r < N$).

Multiplying the expanded constraint Eqs. (2) by Λ_{ik} yields

$$[F^i_n x_n + (1/2!)F^i_{mn} x_m x_n + \cdots]\Lambda_{ik} = 0,\ k = 1,2, \cdots ,S. \tag{6}$$

On multiplying out, the first product vanishes by virtue of (5), leaving in the left hand side the sum of S polynomials whose leading terms are quadratic forms in x_n. Kötter's second-order analysis involves establishing the existence (or otherwise) of a nontrivial set of displacements x_n that, besides satisfying linearized Eq. (3), also turns into zero all of the S quadratic forms; inexistence of such a set is a necessary and sufficient criterion of a first-order infinitesimal mechanism. An example of this analysis can be found in (Kuznetsov, 1993).

There are two problems with this analysis. First, it may be inconclusive, indicating either higher-order infinitesimal, or finite mobility; distinguishing between the two requires higher-order analysis. Second, only in the simplest case of $S = 1$, prestressability (or otherwise) of the system is obvious; it follows from sign definiteness of the only quadratic form present. Bringing into consideration the concept of structural stability changes the situation completely and favorably.

First, recall that all theoretical systems (including prestressable ones) with higher-than-first order of infinitesimal mobility in reality either turn into the respective ordinary type or, if prestressed, possess only first-order mobility. Hence, in the context of real systems, second-order analysis is conclusive and higher-order analysis is not necessary. Second, prestressability, the decisive issue of second-order analysis, lends itself to a straightforward resolution: a necessary and sufficient condition for prestressability is the existence of a set of constraint reactions Λ_{ik} producing a sign-definite combination of the S quadratic forms in (6), subject to relations (3). This is, in fact, the condition originally proposed by Calladine and Pellegrino (1991) for first-order infinitesimal mechanisms; it was later refined (Calladine and Pellegrino, 1992) to accommodate systems not admitting a sign-definite combination of quadratic forms, yet kinematically immobile. Such a system is characterized by the absence of a displacement set x_n satisfying relations (3) and simultaneously turning into zero all S quadratic forms in (6). Although such systems [unprestressable infinitesimal mechanisms; Fig. 3 in (Kuznetsov, 1993)] exist, they are nongeneric and unrealizable. Thus, the existence of a sign-definite combination of the S quadratic forms is the analytical criterion for real systems with first-order mobility.

Turning to quasi-invariant and quasi-variant systems with unilateral constraints, note that their existing analytical criteria require only prestressability, necessary for constraint counteraction. In real systems, actual prestress must be present to ensure constraint engagement.

Finally, the necessary and sufficient criterion for a real geometrically variant system with bilateral constraints is

$$r = C < N. \tag{7}$$

In mathematical terms, real variant systems must be qualified as structurally stable and, in the absence of elastic strains induced by external loads, do not admit singular configurations at all. Singular configurations ($r < C$) are either prestressed, and then quasi-variant; or nongeneric and unrealizable, regardless of whether they formally possess or lack kinematic mobility.

For a real variant system involving unilateral constraints, the following observation holds true: an unprestressed system that includes a wire or a strut and is not geometrically invariant on the account of bars alone, is variant.

In summary, the difference between the presented criteria for real structural systems and the existing formal criteria stems from the distinction between prestressability and prestress. Prestress is sufficient for kinematic immobility of any real system and is also necessary for quasi-invariant and quasi-variant systems, as well as for invariant systems with unilateral constraints. Prestress overrides geometric imperfections, such as lack of precision in the member sizes, thermal distortions, etc. With prestress, the configuration singularity is engendered not by the infeasible exact geometry, but by statics, thus producing a singular configuration somewhat different from the nominal one. The finite elastic strain induced by prestress makes the singularity structurally stable, i.e., generic (at least, locally).

In the mathematical literature (e.g., Connelly and Whiteley, 1996) the term 'prestress stability' is consistently employed. Prestress usually means a stress induced prior to loading, so that its very existence already implies stability. In contrast, self-stress is only a formal, statically possible, self-equilibrated stress state (a solution to the homogeneous equilibrium equations). Self-stress can be proper or improper for the unilateral constraints, and proper self-stress, in turn, can be stable or unstable. Only proper and stable self-stress is physically realizable as prestress. Thus, self-stress stability entails 'prestressability', i.e., the ability of the system to acquire prestress. The term 'prestress stability' also obscures the distinction between mere prestressability and the actual presence of prestress. This subtle distinction is crucial in the analysis of real systems, due to the role of prestress and the accompanying elastic strain as conditions for structural stability.

To sum up, a generically singular configuration of a real structural system is prestressed and kinematically immobile ($V > 0$, $S > 0$, $K = 0$). Unprestressed and, the more so, unprestressable singular configurations, mobile or immobile, are nongeneric, unrealizable, and noncomputable. In what follows, a few interesting implications of this conclusion are presented.

4. Singular configurations with simultaneous statical and kinematic indeterminacy

A structural system with Jacobian matrix rank $r < N$ is underconstrained and allows virtual displacements (it is virtually indeterminate, $V > 0$). For this system to have only virtual, but no finite, kinematic, mobility (i.e., to have a unique geometric configuration), it must a) be singular ($r < C$), hence, statically indeterminate ($S > 0$) with a comprehensive self-stress, and b) admit a sign-definite combination of the S quadratic forms, as discussed earlier. Without satisfying condition b), the configuration is singular but the system is kinematically indeterminate ($K > 0$). The singularity is only local, with $V > K > 0$, if an adjacent configuration is nonsingular (the rank restores to $r_{\max} = C$, leading to $V = K > 0$ and recovered statical determinacy). If the rank does not restore, the singularity is global, with $S > 0$ and $V = K > 0$ in any kinematically possible configuration; such a system possesses global simultaneous statical and kinematic indeterminacy and is a statically indeterminate finite mechanism.

The above conventional reasoning is to be refined in light of the notion of structural stability. Recall that kinematically mobile configurations are unprestressable (prestressability and kinematic mobility are mutually exclusive). This rules out structural stability, meaning that the above singular configurations are nongeneric, unrealizable, and noncomputable, and so are the systems ostensibly admitting such configurations.

Systems with simultaneous global statical and kinematic indeterminacy were investigated by several authors: Tarnai (1980); Pellegrino and Calladine (1986). Perhaps, the simplest possible example of such a system is a rigid beam with three parallel, equal length, support bars (Fig. 1a) As a result of even the slightest geometric imperfection, the real system reverts to one of the two possible generic types—stress-free invariant (Fig. 1b) or prestressed quasi-invariant (Fig. 1c). The latter requires the three support bar

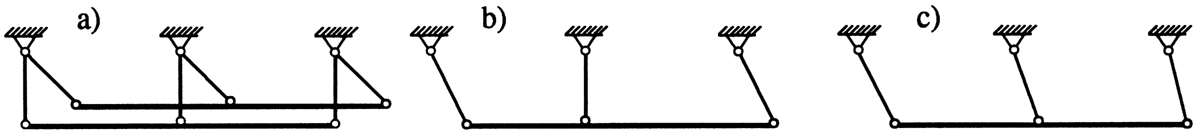

Fig. 1. Theoretical statically indeterminate finite mechanism (in reality, a mechanism with elastic interference): (a) unrealizable, nongeneric singular configuration; (b) ordinary, invariant configuration; (c) generic singular (prestressed quasi-invariant) configuration with three support bar directions intersecting at one remote point.

directions intersecting at one point; other than that, the geometries (Fig. 1b and Fig. 1c) of the two alternative generic configurations can be very similar. In either case, the real system behaves as a finite mechanism with elastic interference (Kuznetsov, 1991): very large displacements are possible at the expense of small elastic strains. Specifically, these small strains are commensurate with the geometric imperfections of the system, and the strain variations in the process of motion are the source of the elastic interference.

Not anticipating such behavior may complicate either of the two alternative applications of this kind of system. As a structure, it is flimsy, with only elastic interference, rigidity of the joints, and friction restraining its mobility; this appears to be the case with the timber octagon of Ely cathedral (Tarnai, 1986). Usually the intended application is a mechanism. Some ingenious mechanisms (foldable structures comprised of angulated rods joined by scissor hinges) have been discovered by You and Pellegrino (1997); their paper also contains an extensive bibliography. However, aside from the expected (and controllable) friction, the assembly process, initial deployment, and mechanical performance of such systems should be strongly affected by their statical indeterminacy. The latter is the source of the elastic interference caused by any departure from the nominal geometry (due to imperfect manufacturing, thermal distortions and even elastic deformations induced by loading). Yet, many applications described in technical publications and patents are suggested without noticing that the proposed mechanisms are statically indeterminate, sometimes to a high degree. When evaluating the elastic interference, initial imperfections can be accounted for in an analysis reflecting their random distribution, whereas the contribution of thermal distortions and elastic deformations due to external loads can be evaluated deterministically.

Turning to singular, hence, statically indeterminate configurations of kinematically mobile systems, recall that these are structurally unstable and unrealizable. An interesting example of this kind of nongeneric singular configuration is a cusp mechanism of Connelly and Servatius (1994), a geometric construct ostensibly possessing higher-order rigidity. In fact, it is a statically indeterminate, hence, singular configuration of a finite mechanism. Although infeasible in a real system, this configuration exhibits some remarkable theoretical features, revealing, in particular, difficulties with the introduced definition of higher-order rigidity. Note that the system has been devised using symbolic (algebraic) calculations, which obscures the fact that it is noncomputable. However, a statement following this example in the paper appears more controversial; it reads: 'the usual definition of N-th order rigidity may be carried over to arbitrary systems of algebraic equations'. But higher-order rigidity for a system of nonlinear algebraic equations is conceivable only as a nested (higher-codimension) singularity which, at least in the context of structural frameworks, is nongeneric, hence, noncomputable. Accordingly, numerical solutions intended to establish the order of rigidity of algebraic systems are doomed. And for the very few algebraic systems admitting symbolic or exact solutions, the latter would still be nongeneric.

This is a good opportunity for correcting an error by the present author in implementing Koiter's

(Koiter, 1984) idea for constructing higher-order infinitesimal mechanisms. The two example systems shown in Fig. 4 and Fig. 5 of (Kuznetsov, 1999) are very similar, and so are the accompanying explanations. The local center of curvature, O, of the path of the top bar midpoint (depicted in Fig. 4 of (Kuznetsov, 1999)) is shown correctly in the Figure. However, in Fig. 5 of (Kuznetsov, 1999) this center of curvature was mistakenly identified with the instant center of rotation of the bar.

5. Polyhedral trusses

A classical object of statical-kinematic analysis is a general polyhedral surface, traditionally represented by one of the two interrelated but distinct models. A continuous (surface) model is an assembly of flat polygonal faces connected by rigid rectilinear ('piano') hinges at the edges. Faces are usually idealized such that their planarity is not enforced (a rigid plate would be an exceedingly restrictive model). The second, discrete, model is a hinge-bar assembly (a polyhedral truss) where rigid bars (edges) are joined by spherical hinges at the vertices; polyhedral face is not an entity in this model. The two models are identical in the case of triangulated polyhedral surfaces, since replacing bar triangles with rigid triangular faces and introducing linear hinges along the edges does not affect the system kinematics.

Rigidity (kinematic immobility) of convex polyhedral surfaces was established by Cauchy. Attempts to improve this result met with limited success and centered on the general conjecture on rigidity of polyhedra. Bricard (1897) discovered a kinematically mobile (in mathematical terms, flexible) octahedron; it is, however, not only nonconvex, but also self-intersecting. A closer look at the hinge-bar (truss) model of Bricard's octahedron reveals that it is a combination of two identical finite mechanisms moving in accord in one common motion. A simple conceptual example of this kind of assembly involves two identical pin-bar chains connected with vertical bars (Fig. 2). The system attributes are: $N=C=4$; $r=3$; $S=V=K=1$, and the Jacobian matrix rank is not affected by the kinematic motion wherein all of the bars undergo rigid-body translations and rotations. Thus, the system is singular, with simultaneous global statical and kinematic indeterminacy. Self-stress in the system comprises a pattern of tension forces in the upper chord, the identical pattern of compression in the bottom chord, and tension in the two posts. The state of self-stress is configuration-dependent, i.e., differs from one configuration to another. The system is nongeneric, structurally unstable, physically unrealizable, and noncomputable.

Bricard's octahedron shares these qualifications. Its truss model shown in Fig. 3 involves two identical hinge-bar pyramids (Fig. 3a, b), with the second pyramid obtained by rotating the first one 180° about a vertical axis. The pyramids are assembled on a common square base (Fig. 3c) but two bars in the model

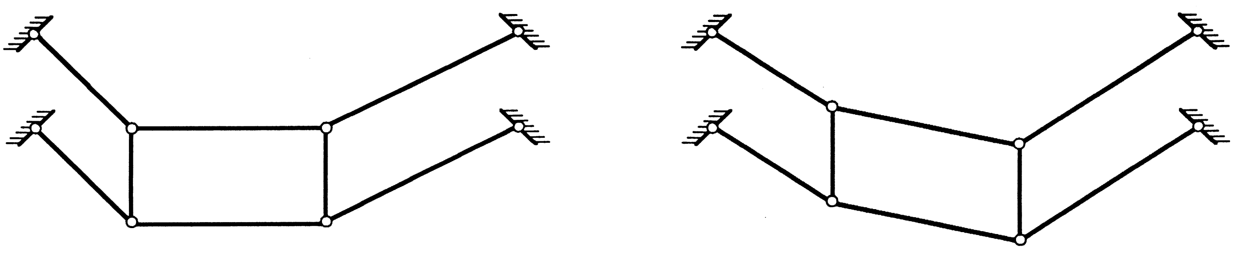

Fig. 2. Kinematically mobile system comprised of two connected finite mechanisms moving in accord. All bars undergo rigid-body translations and rotations.

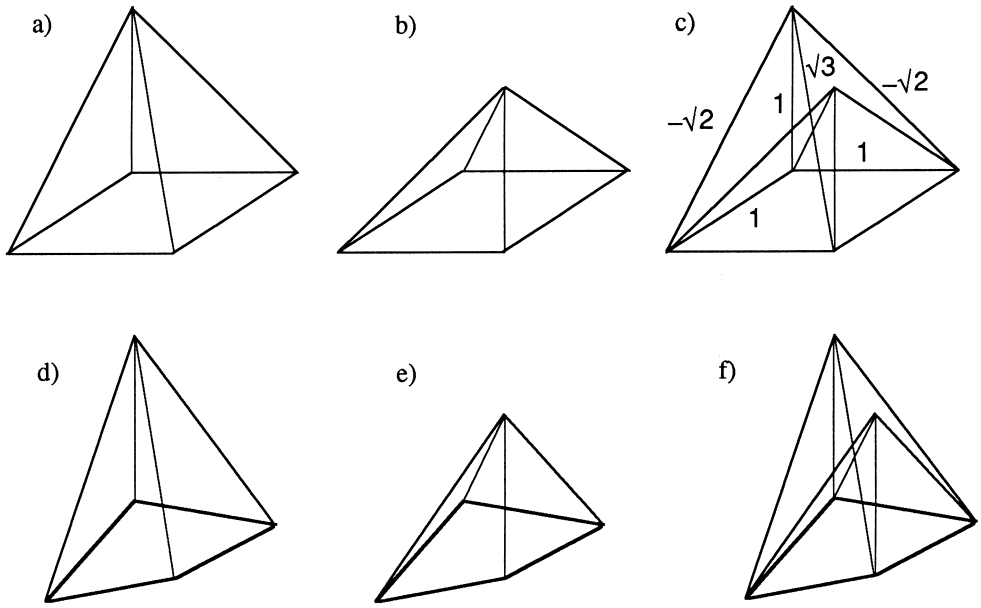

Fig. 3. Composition and kinematics of Bricard's octahedron: (a) pyramid with square base; (b) identical pyramid after 180 degree rotation; (c) two pyramids assembled on common base; (d) and (e) identical deformations of two pyramids; (f) deformation of assembly.

are intersecting. This can be avoided by slightly bending one or both of them: a rigid bent bar still preserves the distance between its end points, hence, is kinematically identical to a straight bar. The kinematic motions of the two pyramids are identical (Fig. 3d, e), so that, when assembled, they move in accord (Fig. 3f). Assuming the system properly supported in three-dimensional space, its attributes are: $N = 3 \times 6 - 6 = 12 = C$; $r = 11$; $S = V = K = 1$. Self-stress exists for any configuration of the system, and its pattern is antisymmetric with respect to one of the vertical planes of symmetry; shown in Fig. 3c are the self-stress forces for one half of the system in its original configuration (the remaining forces are obtained by sign reversal).

The surface model of Bricard's octahedron is triangulated (the base square is not a face), hence, kinematically equivalent to the truss model. It is, however, badly self-intersecting and, because of that, disqualified as a counterexample to the rigidity conjecture. For non-intersecting surfaces, the next important result was obtained by Gluck (1975), who proved that almost all simply connected triangulated polyhedral surfaces are rigid. (Mathematically, 'almost' exempts a set of measure zero from the original set). It is interesting to explore this result in the light of the concept of structural stability and taking advantage of the hinge-bar (polyhedral truss) model.

Two relations characterizing simply connected polyhedra are relevant here. One is Euler's equation, $V - E + F = 2$, relating the number of vertices, edges, and faces. The other, $2E \geq 3F$, relates the numbers of edges and faces, with the border equality holding for triangulated polyhedra. Accordingly, the Maxwell number, which is the number of internal degrees of freedom (those associated with the system distortions, as opposed to its rigid motion) for a properly supported polyhedral truss is

$$M \equiv 3V - E - 6 \geq 0 \tag{8}$$

From here it follows immediately that a polyhedral truss with just one non-triangular face has $M \geq 0$, hence, is underconstrained. As a result, almost all such trusses, specifically, those in ordinary configurations (full-rank Jacobian) are kinematically mobile, i.e., geometrically variant. The exceptional, rank-deficient configurations are nongenerically singular, therefore unrealizable; however, if prestressed, they become generic, quasi-variant.

By virtue of $M = 0$, a simply connected triangulated polyhedral truss in an ordinary configuration is statically, virtually and kinematically determinate, hence, geometrically invariant and structurally stable. This conclusion holds, in particular, for polyhedral trusses with triangulated coplanar panels. Thus, almost all simply connected triangulated polyhedral trusses, as well as the underlying surfaces (due to the model equivalence) are infinitesimally rigid, which is the above mentioned result of Gluck (1975).

The exceptional, rank-deficient configurations are singular and possess statical, virtual or, perhaps, even kinematic indeterminacy, the latter depending on whether the singularity is local or global. Only a local singularity can be generic and then only in the expanded state space involving elastic strains of prestress. Since elasticity is not considered in geometric rigidity studies, generic singularity and structural stability are ruled out. Hence, all locally and, the more so, globally singular polyhedral trusses, like Bricard's octahedron, are nongeneric, physically unrealizable, and noncomputable.

Strictly speaking, the statement 'almost all simply connected triangulated surfaces are rigid' is, as a matter of principle, equivalent to 'almost all pencils cannot stand on the sharpened end' or, closer to the current subject, 'almost all polyhedra cannot stand on one vertex upon a plane'. The latter two, perfectly logical, statements are not intended to detract from the first one, rigorous and important mathematical statement. Physical realizability may not concern a mathematician who finds intellectual challenge and aesthetic pleasure in creating exceptional out of common and discovering singular among ordinary. In elegant words of Nobel physicist Richard Feynman, 'Science is as much for intellectual enjoyment as for practical utility'. An applied mechanician, on the other hand, is more likely to bear in mind physical reality and, in particular, to let mathematical facts be confronted with the principle of structural stability.

6. Polyhedral surfaces

After Gluck's (1975) advance, the final verdict on the general rigidity conjecture depended on the kinematic properties (mobility or otherwise) of the singular surfaces exempted by his theorem. The issue was brought to an unexpected closure in 1977 by R. Connelly. His breakthrough result, elaborately presented in Connelly (1979), shows that not all triangulated closed surfaces are rigid. In what amounts to a geometric invention, a special construct (a 'crinkle') has been introduced to eliminate self-intersections in Bricard's octahedron without immobilizing it. The obtained simply connected, nonconvex, triangulated polyhedral surface is a counterexample to the rigidity conjecture; it is a 'flexible triangulated sphere'. However, like its precursor, Bricard's octahedron, it is nongenerically singular, physically unrealizable, and noncomputable.

Continuing the foregoing analogy, the statement 'not all triangulated closed surfaces are rigid' is equivalent to 'not all pencils are unable to stand on the sharpened end'. Obviously, a pencil standing on the sharpened end is a statical possibility (it satisfies the pertinent equilibrium equation) but, because of geometric instability, it is unrealizable. The implication for the flexible sphere is analogous: this exact theoretical-geometric construct is geometrically possible (by implicitly satisfying pertinent geometric construction rules) but, being structurally unstable, is unrealizable.

The simplest flexible polyhedron discovered so far has just nine vertices (Connelly, 1979). If constructed, it must revert to one of the two possible generic types—stress-free invariant or prestressed quasi-invariant. Indeed, when assembling cardboard models of this polyhedron, joining the last edge seams has always required a noticeable effort (the act of prestressing!). Alternatively, the shapes of the faces sharing the last seams to be closed, could be adjusted for an effortless assembly of an invariant (but nearly singular, hence, elastically flexible) polyhedron. Assembled models consistently exhibit the expected structural behavior: they develop appreciable resistance to any displacement, revealing the elastic resistance unavoidable in either of the two real embodiments. The described outcomes for this theoretical system with simultaneous statical and kinematic indeterminacy can be once again traced back to the system in Fig. 1.

Discrete hinge-bar model can be adapted for kinematic analysis of general (not triangulated) polyhedral surfaces by incorporating continuous two-dimensional members for the face panels. In this model a panel is represented by a very thin plate capable of resisting in-plane tension, compression and shear, but devoid of resistance to bending. Such a panel preserves the intrinsic geometry of the face (in-plane distances and angles), but allows out-of-plane bending and warping. In particular, non-smooth bending in the form of creasing is allowed, but creases cannot intersect, branch or bend. [This is in contrast with a unilateral (soft) membrane that supports only tension and allows irregular bending (wrinkling) not preserving the intrinsic geometry of the face.] A plate panel deprives the peripheral edge n-gon of all of its $2n - n - 3 = n - 3$ in-plane degrees of freedom, whereas out-of-plane mobility of the face is not affected.

Let n_f ($f = 1, 2,..., F$) be the number of sides in the f-th face of a simply connected polyhedron. Since each of the E edges in the polyhedron is shared by two faces, summing up the number of sides in all faces gives $\Sigma\ n_f = 2E$. Accordingly, the total number of in-plane degrees of freedom taken away by the combined effect of all F face panels is

$$\Sigma(n_f - 3) = \Sigma n_f - 3F = 2E - 3F, \quad f = 1,2,\cdots,F. \tag{9}$$

The number of internal (distortion producing) degrees of freedom for the model is obtained by subtracting the above number from the Maxwell number of the corresponding polyhedral truss. In view of the Euler formula, the resulting Maxwell number for the considered model is

$$M = 3V - E - 6 - (2E - 3F) = 0. \tag{10}$$

Thus, in any ordinary configuration (full-rank Jacobian), the described discrete model of a general polyhedron is virtually (hence, also kinematically) determinate, geometrically invariant and structurally stable. A somewhat more general conclusion is that invariant intrinsic geometry of faces is both necessary and sufficient for geometric invariance (infinitesimal rigidity) of a simply connected polyhedral surface in a non-singular configuration.

Two remarks are in order.

1. The Maxwell number in (10) reflects the system kinematics but does not indicate statical determinacy. In fact, the above panel model of a polyhedral face is statically indeterminate.
2. An inextensible, soft (wrinkling) membrane attached to the peripheral bar polygon of a face is prestressable if the face is convex, and then it preserves the intrinsic geometry of the face. However, this model is nongeneric unless the membrane is actually prestressed.

7. Conclusions

1. Structurally unstable (nongeneric) models are formal constructs of purely theoretical value. They are physically unrealizable and noncomputable; their implementation would produce a generic prototype with essentially different properties and structural behavior.
2. The only real (physically and numerically realizable) types of structural system are the two ordinary

types (geometrically invariant or variant) and their generically singular configurations (quasi-invariant or quasi-variant); the latter two have prestress of finite magnitude and possess first-order infinitesimal mobility.

3. Singular configurations of structural systems with ideal constraints are nongeneric. They become generic for elastic systems and then are realizable by statical (but not geometric) means, specifically, by prestress producing elastic strains of finite magnitude.
4. Unprestressed and, the more so, unprestressable singular configurations are nongeneric, unrealizable and noncomputable. The case of exact or symbolic (algebraic) calculations for such models may obscure the fact that they are just formal analytical constructs.
5. Complying with the concept of structural stability streamlines the analytical criteria for the four types of real structural systems and leads to a simpler (second-order), yet always conclusive, statical-kinematic analysis for real systems.
6. Simultaneous statical and kinematic indeterminacy is impossible in real structural systems. The feasible combinations of statical-kinematic properties for (load-free) real systems are:
   a) statically determinate systems are always stress-free, whereas statically indeterminate systems are never stress-free;
   b) a system can be simultaneously statically and kinematically determinate ($r=N=C$), whereas the two indeterminacies are mutually exclusive; specifically,
   c) statically indeterminate systems are kinematically determinate (immobile) and
   d) all configurations of kinematically indeterminate systems are statically determinate.
7. A simply connected polyhedral surface in a nonsingular configuration is geometrically invariant (infinitesimally rigid) if, and only if, intrinsic geometry of the faces is preserved.
8. A real simply connected polyhedral surface is kinematically immobile: stress-free, invariant (first-order rigid) in an ordinary configuration or prestressed, quasi-invariant (second-order rigid) in a singular configuration.
9. Kinematically mobile (flexible) simply connected polyhedral surfaces are nongenerically singular geometric constructs, thus, physically and numerically unrealizable.

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